Recursive SMLR deconvolution algorithm for Bernoulli–Gaussian signals

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Abstract: In the past decade, many detection and estimation algorithms have been reported for estimating a desired Bernoulli–Gaussian signal which was distorted by a linear time-invariant system. The well known Kormylo and Mendel's single most likely replacement (SMLR) algorithm, which works well and has been successfully used to process real seismic data, is an offline signal processing algorithm. The paper proposes a recursive SMLR algorithm which has online data processing capabilities and requires much less computational effort than Chi and Mendel's recursive algorithm and Goussard and Demoment's recursive algorithm. Simulation results show good performance.

1 Introduction

Estimating a desired signal $\mu(k)$ from a given set of noisy data $z(k)$ based on the following convolutional model

$$z(k) = \mu(k) * v(k) + n(k) = \sum_{i=0}^{\infty} v(i)\mu(k-i) + n(k) \quad (1)$$

is a deconvolution problem, where $n(k)$ is measurement noise and $v(k)$ is the impulse response of a linear time-invariant signal distorting system which corresponds to such examples as the source wavelet in seismic deconvolution [1–3] and the channel impulse response in channel equalisation [4] (in communications). The convolutional model eqn. 1 can also be represented in an nth-order state-variable form, as

$$\boldsymbol{x}(k) = \Phi\boldsymbol{x}(k-1) + \gamma\mu(k) \quad (2)$$

$$z(k) = \boldsymbol{h}'\boldsymbol{x}(k) + n(k) \quad (3)$$

where $\boldsymbol{x}(k)$, γ, and $\boldsymbol{h}$ are $n \times 1$ vectors, Φ is an $n \times n$ matrix. Of course, given $v(k)$, there exist many $(\Phi, \gamma, \boldsymbol{h})$'s [5]. Kormylo and Mendel [6] proposed a Bernoulli–Gaussian (B–G) model, which has been used in seismic deconvolution and biomedical ultrasonic imaging, for a sparse spike sequence as

$$\mu(k) = r(k) \cdot q(k) \quad (4)$$

where $r(k)$ is a white Gaussian random sequence with variance σ_r^2 and $q(k)$ is a Bernoulli sequence for which

$$P_r[q(k)] = \begin{cases} \lambda, & q(k) = 1 \\ 1-\lambda, & q(k) = 0 \end{cases} \quad (5)$$

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Assuming that $n(k)$ is zero mean white Gaussian with variance σ_n^2 and based on the state-variable model (eqns. 2 and 3), they developed a suboptimal maximum-likelihood (ML) algorithm [2, 3, 6], called the single most likely replacement (SMLR) algorithm for estimating $\mu(k) = r(k) \cdot q(k)$ using measurements $z(1), z(2), \ldots, z(N)$. This SMLR algorithm works well and has been successfully used to process real seismic data [3, 7].

Chi *et al.* [8] and Giannakis *et al.* [9] reported computationally fast suboptimal ML algorithms for estimating B–G signals. However, these algorithms are offline signal processing algorithms. As mentioned in Reference 10, online signal processing is needed when large amounts of data are associated with real-time constraints and limited computational power. For example, in biomedical ultrasonic imaging, routine nondestructive evaluation, and on-site seismic data processing. Several recursive algorithms for online signal processing were also reported and can be found in the literature [10–14]. In this paper, we propose a recursive SMLR algorithm which has online data processing capabilities and inherits all the performance advantages of the offline SMLR algorithm, assuming that $v(k)$ and statistical parameters λ, σ_r^2 and σ_n^2 are given *a priori*.

2 Recursive SMLR algorithm

The proposed recursive SMLR algorithm is basically a fixed-lag signal processing algorithm. A block of $z(i)$, $i = 1, 2, \ldots, L$, where L is the block size, is processed to yield $\hat{\mu}(1)$. Then the next block of $z(i)$, $i = 2, 3, \ldots, L+1$, is processed to yield $\hat{\mu}(2) \cdot \hat{\mu}(i)$ for $i \geqslant 3$ are obtained so on and so forth.

Let $\boldsymbol{z}_k = (z(1), z(2), \ldots, z(k), \ldots, z(k+L-1))'$ and $\boldsymbol{q}_k = (q(1), q(2), \ldots, q(k), \ldots, q(k+L-1))'$. The ML estimate $\hat{\boldsymbol{q}}_k$ is the one that maximises the likelihood function

$$S_k\{\boldsymbol{q}_k | \boldsymbol{z}_k\} = p(\boldsymbol{z}_k, \boldsymbol{q}_k) \quad (6)$$

when $\boldsymbol{q}_k = \hat{\boldsymbol{q}}_k$. Notice that S_k includes not only past and present measurements $z(1)$ through $z(k)$ but also 'future' measurements $z(k+1)$ through $z(k+L-1)$. Let $\hat{q}(i)$ denote the 'optimum' estimate of $q(i)$, associated with S_i. The proposed recursive algorithm detects $q(k)$ based on S_k by searching for the optimum $q(k)$ through $q(k+L-1)$ with $q(1) = \hat{q}(1)$, $q(2) = \hat{q}(2)$, $\ldots$, $q(k-1) = \hat{q}(k-1)$.

The iterative offline SMLR algorithm reported in Reference 6, with some modifications, fits the need for the proposed recursive SMLR algorithm at each recursion. Let $\Lambda(j)$ denote the likelihood ratio

$$\Lambda(j, q_r) = \frac{S_k\{\boldsymbol{q}_j | \boldsymbol{z}_k\}}{S_k\{\boldsymbol{q}_r | \boldsymbol{z}_k\}} \quad k \leqslant j \leqslant k+L-1 \quad (7)$$

where $\boldsymbol{q}_r = (q_r(1) = \hat{q}(1),\ q_r(2) = \hat{q}(2),\ \ldots,\ q_r(k-1) = \hat{q}(k-1),\ q_r(k),\ q_r(k+1),\ \ldots,\ q_r(k+L-1))'$ is a reference sequence and $\boldsymbol{q}_j$ is a test sequence defined as

$$q_j(i) = \begin{cases} q_r(i) & i \neq j \\ 1 - q_r(i) & i = j \end{cases} \tag{8}$$

which differs from $\boldsymbol{q}_r$ only at a single time location j. During each iteration, the recursive SMLR algorithm searches for the 'optimal' $q(k)$ through $q(k+L-1)$ as follows:

(*a*) Compute $\ln \Lambda(j)$ for $j = k, k+1, \ldots, k+L-1$.

(*b*) Assume that $\ln \Lambda(j') = \max\{\ln \Lambda(j),\ k \leqslant j \leqslant k+L-1\}$; if $\ln \Lambda(j') > 0$, update $q_r(j')$ by $1 - q_r(j')$ and go to (*a*).

When $\ln \Lambda(j) \leqslant 0$ for all $k \leqslant j \leqslant k+L-1$, the detection of $q(k)$ is finished and the obtained $\hat{q}_r(k)$ is the desired estimate, $\hat{q}(k)$, of $q(k)$. The by-product $q_r(i)$, $k+1 \leqslant i \leqslant k+L-1$ together with either $q(k+L) = 0$ or $q(k+L) = 1$ can be used as the initial conditions at next recursion associated with S_{k+1}. When $k = 1$ and $L = N$ (total number of data), the recursive SMLR algorithm reduces to the offline SMLR algorithm. It is advisable here to indicate the distinctions between the recursive SMLR algorithm and some other recursive suboptimal ML algorithms.

The recursive SMLR algorithm differs from Kormylo's recursive algorithm [12] in that the latter detects $q(k)$ based on S_k by letting $q(1) = \hat{q}(1)$, $q(2) = \hat{q}(2)$, ..., $q(k-1) = \hat{q}(k-1)$ and $q(i) = E[q(i)] = \lambda$ for $k+1 \leqslant i \leqslant k+L-1$, which is not consistent with the B–G assumption for $q(i)$, $k+1 \leqslant i \leqslant k+L-1$. Goussard and Demoment's recursive algorithm [10] detects $q(k)$ based on the expected value of S_k over all possible $q(i)$'s for $k+1 \leqslant i \leqslant k+L-1$ with $q(1) = \hat{q}(1)$, $q(2) = \hat{q}(2)$, ..., $q(k-1) = \hat{q}(k-1)$. Chi and Mendel's Viterbi algorithm [11] basically detects $q(k)$ by letting $q(1) = \hat{q}(1)$, $q(2) = \hat{q}(2)$, ..., $q(k-1) = \hat{q}(k-1)$ and searching for the maximum S_k over all possible $q(i)$'s for $k \leqslant i \leqslant k+L-1$.

Next, we turn to the computational issue of the recursive SMLR algorithm. $\ln \Lambda(j)$ has been shown [2, 6] to be

$$\ln \Lambda(j, \boldsymbol{q}_r) = \frac{1}{2}\frac{\sigma_r^2 f_j^2 \rho_j}{1 + \sigma_r^2 a_j \rho_j} - \tfrac{1}{2}\ln(1 + \sigma_r^2 a_j \rho_j) + \rho_j \ln\frac{\lambda}{1-\lambda} \tag{9}$$

where

$$f_j = \boldsymbol{v}_j \Omega_r^{-1} \boldsymbol{z}_k \tag{10}$$

$$a_j = \boldsymbol{v}_j' \Omega_r^{-1} \boldsymbol{v}_j \tag{11}$$

$\rho_j = 1 - 2q_r(j)$, $\boldsymbol{v}_j = (0, 0, \ldots, v(0), v(1), \ldots, v(k-j))'$ and $\Omega_r = E[\boldsymbol{z}_k \boldsymbol{z}_k' \,|\, \boldsymbol{q} = \boldsymbol{q}_r]$. f_j and a_j for $k \leqslant j \leqslant k+L-1$ can be obtained by running a Kalman filter type optimal smoother [2, 6] based on the state-variable model eqns. 2 and 3 with the initial conditions $\hat{\boldsymbol{x}}(k-1\,|\,k-1)$ and $P(k-1\,|\,k-1)$, where $\hat{\boldsymbol{x}}(k-1\,|\,k-1)$ is the filtered estimate of $\boldsymbol{x}(k-1)$ and $P(k-1\,|\,k-1)$ is the associated error covariance matrix and both of them are available prior to time point k. The reader can refer to References 2 and 6 for the detailed procedure of computing f_j and a_j for $k \leqslant j \leqslant k+L-1$. Therefore, the computational load for detecting $q(k)$ is approximately only two Kalman filters (one optimal smoother) running over L observations multiplied by the number, l, of iterations of running (*a*) and (*b*) at each recursion. In general, $l \ll 2^L$ which implies a great computational saving compared with Chi and Mendel's VA algorithm [11] and Goussard and Demoment's recursive algorithm [10] since both of them need 2^L Kalman filters running over L data associated with all possible $q(j)$'s for $k \leqslant j \leqslant k+L-1$ at each recursion. The proposed recursive SMLR algorithm is therefore a computationally efficient algorithm and inherits all performance advantages of the well known offline SMLR algorithm.

If $\hat{q}(k) = 0$, we proceed with the next recursion associated with S_{k+1}. When $\hat{q}(k) = 1$, we then have to estimate $r(k)$. It is well known that the ML estimate $\hat{r}(k)$ [2] is given by

$$\hat{r}(k) = \sigma_r^2 q(k) f_k \tag{12}$$

Therefore, obtaining $\hat{r}(k)$ using eqn. 12 with $q(k) = \hat{q}(k)$ is trivial.

3 Computer simulations

We generated noise free data by convolving a selected wavelet $v(k)$, taken from Reference 2 and shown in Fig. 1,

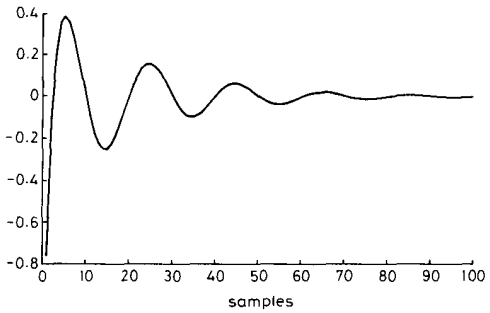

Fig. 1 *Selected wavelet $v(k)$*

with a B–G signal with parameters $\lambda = 0.1$, $\sigma_r^2 = 1$. We then added a pseudo-Gaussian noise sequence to the noise free data to form the synthetic data $z(k)$ shown in Fig. 2

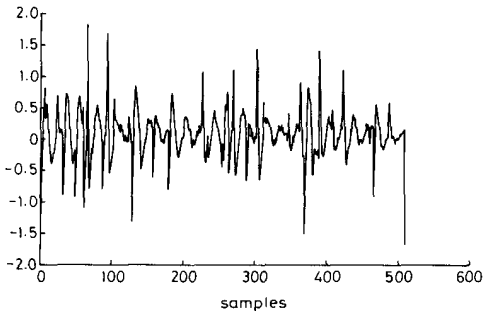

Fig. 2 *Synthetic data*

with a signal-to-noise ratio (SNR) equal to five. We then estimated $\mu(k)$ using the recursive SMLR algorithm. We also estimated $\mu(k)$ using the offline SMLR algorithm for comparison. The deconvolved results are shown in Fig. 3 where circles denote true spikes and bars denote estimates.

The results for L equal to 2, 6 and 10 are shown in Figs. 3*a–c*, respectively. The results using the offline SMLR algorithm for $N = 512$ are shown in Fig. 3*d*. Apparently, the results shown in Fig. 3*b* are inferior to those shown in Fig. 3*c*, and are much better than those shown in Fig. 3*a*, from which one can observe that quite

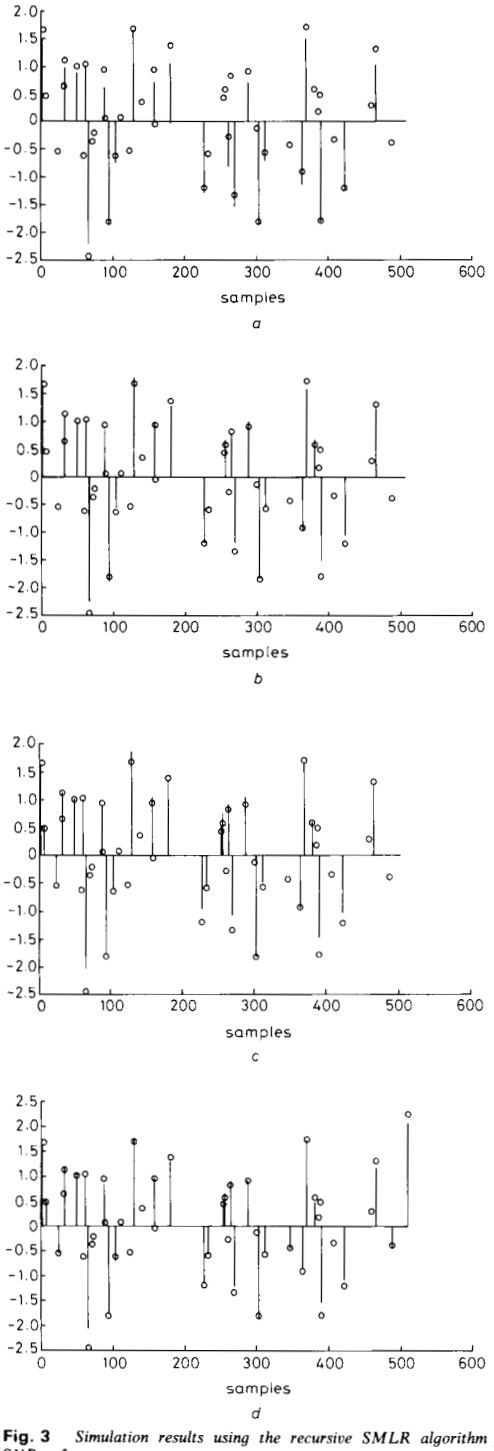

Fig. 3 *Simulation results using the recursive SMLR algorithm for SNR = 5*

Circles denote true spikes and bars denote estimates
a $L = 2$
b $L = 6$
c $L = 10$
d Corresponding results using the off-line SMLR algorithm

a few moderate spikes were missed. The fact that the recursive SMLR algorithm performs better for a larger L is consistent with the fact that more 'future' observations improve its performance. The results shown in Fig. 3*d* are better than those shown in Figs. 3*a–c*. Notice that a large spike at the end of $\mu(k)$ in Fig. 3*d* was not included in Fig. 3*a–c* because the last L data of $z(k)$ were not processed by the recursive SMLR algorithm. Nevertheless, the results shown in Fig. 3*c* are comparable to those shown in Fig. 3*d*. This fact indicates that the performance of the recursive SMLR algorithm is satisfactory with $L \geqslant 10$ for this case. No doubt, for the same performance a larger L is needed for a lower SNR. Of course, the cost of computation efforts and memories needed is proportional to L. The selection of L is, therefore, a tradeoff of performance and cost to be determined by the user.

4 Discussion and conclusions

We have presented a recursive SMLR deconvolution algorithm, which has online data processing capabilities and requires much less computational effort than Chi and Mendel's recursive algorithm [11] and Goussard and Demoment's recursive algorithm [10], for estimating a sparse spike sequence $\mu(k)$ modelled as a B–G signal which was distorted by a linear time-invariant system. At each time point, $q(k)$ is detected by maximising S_k given by eqn. 6 under the constraint that $q(i) = \hat{q}(i)$ for $i \leqslant k - 1$ where $\hat{q}(i)$ is the optimum estimate of $q(i)$ associated with S_i. At each recursion, the recursive SMLR algorithm follows the same detection procedure as the well known offline SMLR algorithm with some modifications. A block of data $z(k)$ through $z(k + L)$ is processed to yield $\hat{q}(k)$. Therefore, L-sample time delay is necessary. The performance of the algorithm is better for a larger L at the expense of more computation effort and more memory. Therefore, the determination of L is a tradeoff of performance and cost. Finally, we also showed some simulation results which support the proposed recursive algorithm.

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Erratum

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The last eqn. before eqn. 3 should read

$$\{\chi_k^s\}_{s=1}^{n_k} = \{\chi_{k-1}^i \cap \psi_k^j \cap \Gamma_k^l\}_{i=1,\, j=1,\, l=1}^{n_{k-1},\, M_k,\, m}$$

Eqn. 22 should read

$$\tilde{x}_k = \frac{1}{\beta_k} \sum_{i \in I} \beta_k^i \tilde{x}_k^i$$

Eqn. 23 should read

$$\tilde{P}_k = \frac{1}{\beta_k} \sum_{i \in I} \beta_k^i (\tilde{P}_k^i + \tilde{x}_k^i \tilde{x}_k^{iT}) - \tilde{x}_k \tilde{x}_k^T$$